A dynamic view of the Heston model

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Based on joint works with Antoine Jacquier (Imperial College London):

- The small-maturity Heston forward smile. *SIAM Journ on Fin. Math. (4)1, 2013.*
- Asymptotics of forward implied volatility. *Submitted, arxiv 1212.0779.*
- Large-maturity regimes of the Heston forward smile. *Submitted, arxiv 1410.7206.*

(Spot) implied volatility

- Asset price process: $(S_t = \mathrm{e}^{X_t})_{t\geq 0}$, with $X_0 = 0$.
- No dividend, no interest rate.
- Black-Scholes-Merton (BSM) framework:

$$C_{\mathrm{BS}}(\tau, k, \sigma) := \mathbb{E}_0\left(\mathrm{e}^{X_\tau} - \mathrm{e}^k\right)_+ = \mathcal{N}(d_+) - \mathrm{e}^k\mathcal{N}(d_-),$$

$$d_\pm := -\frac{k}{\sigma\sqrt{\tau}} \pm \frac{1}{2}\sigma\sqrt{\tau}.$$

- Spot implied volatility $\sigma_\tau(k)$: the unique (non-negative) solution to

$$C_{\mathrm{observed}}(\tau, k) = C_{\mathrm{BS}}(\tau, k, \sigma_\tau(k)).$$

- Spot implied volatility: quoting mechanism in option markets and provides a useful metric to compare options with different strikes and maturities.
- However not available in closed form for most models.

Spot implied volatility ($\sigma_\tau(k)$) asymptotics as $|k| \uparrow \infty$, $\tau \downarrow 0$ or $\tau \uparrow \infty$:

- Berestycki-Busca-Florent (2004): small-τ using PDE methods for diffusions.
- Henry-Labordère (2009): small-τ asymptotics using differential geometry.
- Forde et al.(2012), Jacquier et al.(2012): small-and large-τ using large deviations and saddlepoint methods.
- Lee (2003), Benaim-Friz (2009), Gulisashvili (2010-2012), De Marco-Jacquier-Hillairet (2013): $|k| \uparrow \infty$.
- Laurence-Gatheral-Hsu-Ouyang-Wang (2012): small-τ in local volatility models.
- Fouque et al.(2000-2011): perturbation techniques for slow and fast mean-reverting stochastic volatility models.
- Mijatović-Tankov (2012): small-τ for jump models.
- Gerhold-Gülüm (2013): small-τ implied volatility slope for Lévy models

Related works:

- Kim, Kunitomo, Osajima, Takahashi (1999-...) : asymptotic expansions based on Kusuoka-Yoshida-Watanabe method (expansion around a Gaussian).
- Deuschel-Friz-Jacquier-Violante (CPAM 2014), De Marco-Friz (2014): small-noise expansions using Laplace method on Wiener space (Ben Arous-Bismut approach).

Note: from expansions of densities to implied volatility asymptotics is 'automatic' (Gao-Lee (2013)).

Forward implied volatility

- Fix $t > 0$: forward-start date; $\tau > 0$: remaining maturity.
- Forward-start call option is a European call option with payoff

$$\left(\frac{S_{t+\tau}}{S_t} - \mathrm{e}^k\right)^+ = \left(\mathrm{e}^{X_{t+\tau}-X_t} - \mathrm{e}^k\right)^+,$$

  and value today

$$\mathbb{E}_0\left(\mathrm{e}^{X_{t+\tau}-X_t} - \mathrm{e}^k\right)^+.$$

- BSM model: its value today is simply worth $C_{\mathrm{BS}}(\tau, k, \sigma)$ (stationary increments).
- Forward implied volatility $\sigma_{t,\tau}(k)$: the unique solution to

$$C_{\mathrm{observed}}(t, \tau, k) = C_{\mathrm{BS}}(\tau, k, \sigma_{t,\tau}(k)).$$

- Obviously, $\sigma_{0,\tau}(k) = \sigma_\tau(k)$.
- *Alternative definition:* $\left(S_{t+\tau} - S_t\mathrm{e}^k\right)^+$.

Existing literature on forward smiles

- Glasserman and Wu (2011): different notions of forward volatilities to assess their predictive values in determining future option prices and future implied volatility.
- Keller-Ressel (2011): when the forward-start date t becomes large (τ fixed).
- Empirical results: Balland (2006), Bergomi (2004), Bühler (2002), Gatheral (2006).
- Bompis-Hok (2013): expansion in local volatility models with bounded diffusion coefficient.

Heston

In Heston the (log) stock price process is the unique strong solution to the following SDEs:

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}V_t\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t, & X_0 = 0, \\ \mathrm{d}V_t &= \kappa\left(\theta - V_t\right)\mathrm{d}t + \xi\sqrt{V_t}\mathrm{d}Z_t, & V_0 = v > 0, \\ \mathrm{d}\left\langle W, Z\right\rangle_t &= \rho\mathrm{d}t, \end{aligned}$$

with $\kappa > 0$, $\xi > 0$, $\theta > 0$ and $|\rho| < 1$.

Today's menu: model risk analysis

$(S_{1.5} - KS_1)^+$: What is the range of no-arbitrage prices given calibration to the implied volatility smiles for S_1 and $S_{1.5}$?

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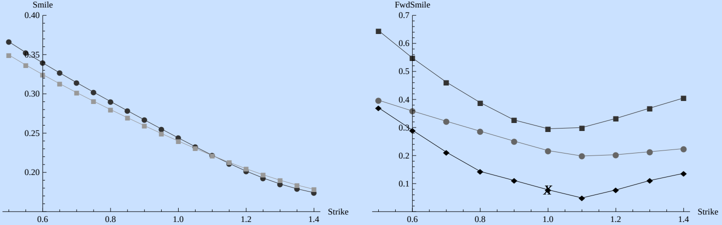
$(S_{1.5} - KS_1)^+$: What is the range of no-arbitrage prices given calibration to the implied volatility smiles for S_1 and $S_{1.5}$? Using martingale optimal transport theory (Henry-Labordère et al.):



(a) Spot Implied Volatility.

(b) Forward Implied Volatility.

Figure: In (a) circles (squares) represents the 1 year (1.5 year) spot implied volatility. In (b) circles plot the Heston forward volatility consistent with the marginals, squares and diamonds plot the lower and upper bounds found by solving the LP problem.

Today's menu: model risk analysis

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Hence, models used for forward volatility dependent exotics should

- have the capability of calibration to liquid forward smiles;
- **produce realistic forward smiles that are consistent with trader expectations and observable prices.**

Today we will develop small and large-maturity forward smile asymptotics in the Heston model and use these results to study the 2nd point.

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The forward smile can also be used as a **metric to understand the dynamics of implied volatility smiles**: Bergomi(2004) calls it a 'global measure' of the dynamics of implied volatilities: Recall $C(t, \tau, k) = \mathbb{E}_0\left[\mathbb{E}\left[(\mathrm{e}^{X_{t+\tau}-X_t} - \mathrm{e}^k)^+ | \mathcal{F}_t\right]\right]$.

We will use our results to shed light on this as well.

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Diagonal small-maturity. Small t and τ : $\Lambda_\varepsilon(u) := \varepsilon \log \mathbb{E}\left(\mathrm{e}^{uX_{\varepsilon\tau}^{(\varepsilon t)}/\varepsilon}\right)$.

Small-maturity. Fixed $t > 0$, and $\tau \downarrow 0$: $\Lambda_\tau^{(t)}(u, \sqrt{\tau}) := \sqrt{\tau} \log \mathbb{E}\left(\mathrm{e}^{uX_\tau^{(t)}/\sqrt{\tau}}\right)$.

Large-maturity. Fixed $t \geq 0$, and $\tau \uparrow \infty$: $\Lambda_\tau^{(t)}(u) := \tau^{-1} \log \mathbb{E}\left(\mathrm{e}^{uX_\tau^{(t)}}\right)$.

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The objective of the analysis is to **relate an expansion of the re-scaled mgf to an expansion of the forward smile**. The nature of the expansion depends critically on the behaviour of the limiting mgf (eg. $\Lambda_0(u) \equiv \lim_{\varepsilon \downarrow 0} \Lambda_\varepsilon(u)$).

Overview of results

If Λ_0 is strictly convex and essentially smooth on its effective domain then

Theorem (Gärtner-Ellis)

If $\Lambda_0(u) := \lim_{\varepsilon\downarrow 0} \Lambda_\varepsilon(u)$ exists in $\mathbb{R} \cup \{+\infty\}$, $0 \in \mathcal{D}_0^o$ and Λ_0 is strictly convex and differentiable on $\mathcal{D}_0^o$, with $\lim_{u\in\partial\mathcal{D}_0} |\Lambda_0'(u)| = +\infty$, then $(X_{\varepsilon\tau}^{(\varepsilon t)})_{\varepsilon>0}$ satisfies a large deviations principle (LDP) (with speed ε) as $\varepsilon \downarrow 0$:

$$\mathbb{P}(X_{\varepsilon\tau}^{(\varepsilon t)} \in A) \sim \exp\left(-\frac{1}{\varepsilon}\inf\{\Lambda^*(x) : x \in A\}\right), \quad A \subset \mathbb{R}, \quad \Lambda^*\text{: dual of } \Lambda_0.$$

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Otherwise (due to extreme moment explosions of $X_\tau^{(t)}$) we enter a number of degenerate large deviations regimes: in these cases a different type of analysis is required and the structure of the asymptotics can look quite different.

Diagonal small-maturity Heston

Here Λ_0 is essentially smooth and strictly convex on its effective domain. The following result is a special case of a much more general result in our paper:

Proposition (Jacquier-R, 2013)

The following expansion holds for the forward smile for all $k \in \mathbb{R}$ as $\varepsilon \downarrow 0$:

$$\sigma^2_{\varepsilon t, \varepsilon \tau}(k) = v_0(k,t,\tau) + v_1(k,t,\tau)\varepsilon + v_2(k,t,\tau)\varepsilon^2 + \mathcal{O}\left(\varepsilon^3\right),$$

where $v_0(\cdot,t,\tau)$, $v_1(\cdot,t,\tau)$ and $v_2(\cdot,t,\tau)$ are continuous functions on $\mathbb{R}$.

In Heston: $\Lambda_\varepsilon(u) = \sum_{i=0}^{2} \Lambda_i(u)\varepsilon^i + \mathcal{O}(\varepsilon^3)$, as ε tends to zero.

- The functions v_0, v_1, and v_2 depend on derivatives of the functions Λ_0, Λ_1 and Λ_2 evaluated at a strike dependent point $u^*(k)$ given as the solution to the equation $\Lambda_0'(u^*(k)) = k$.
- Strict convexity and essential smoothness always ensure that there exists a unique solution to this equation for all $k \in \mathbb{R}$.

Diagonal small-maturity Heston

- We can compare spot and forward (diagonal) small-maturity smiles:

$$\sigma_{\varepsilon t,\varepsilon\tau}(0) = \sigma_{0,\varepsilon\tau}(0) - \frac{\varepsilon t}{8\sqrt{v}}\left(\xi^2 + 4\kappa(v-\theta)\right) + \mathcal{O}(\varepsilon^2),$$

$$\partial_k^2\sigma_{\varepsilon t,\varepsilon\tau}(0) = \partial_k^2\sigma_{0,\varepsilon\tau}(0) + \frac{\xi^2 t}{4\tau v^{3/2}} + \mathcal{O}(\varepsilon).$$

- If the model is calibrated to at-the-money spot volatilities and $v \geq \theta$ then increasing the volatility of variance (or spot convexity) will tend to decrease the at-the-money forward volatility: this is empirically well-known (eg. negative vol of vol dependence for at-the-money forward straddles).
- The relative values of the initial variance v and the mean reversion level θ provide some control on the level of the forward smile vs the spot smile. (eg. if $v \geq \theta$ then $\sigma_{\varepsilon t,\varepsilon\tau}(0) < \sigma_{0,\varepsilon\tau}(0)$)
- At zeroth order in ε the wings of the forward smile increase to arbitrarily high levels with decreasing maturity.

Heston forward smile explosion

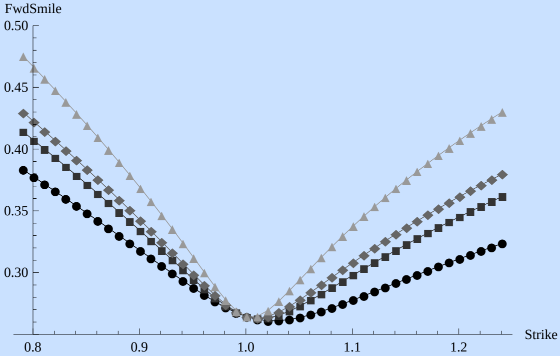

Figure: We plot forward smiles with forward-start date $t = 1/2$ and maturities $\tau = 1/6, 1/12, 1/16, 1/32$ given by circles, squares, diamonds and triangles respectively using the Heston parameters $v = 0.07, \theta = 0.07, \kappa = 1, \rho = -0.6, \xi = 0.5$ and the asymptotic.

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Rescaled log mgf: $\Lambda_\tau^{(t)}(u, a) := a \log \mathbb{E}\left(\mathrm{e}^{uX_\tau^{(t)}/a}\right)$.

Lemma

Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous function such that $\lim_{\tau\downarrow 0} h(\tau) = 0$ and $a \in \mathbb{R}_+^*$.

(i) If $h(\tau) \equiv a\sqrt{\tau}$ then $\lim_{\tau\downarrow 0} \Lambda_\tau^{(t)}(u, h(\tau)) = 0$, for $|u| < a/\sqrt{\beta_t}$ and ∞ otherwise;

(ii) if $\sqrt{\tau}/h(\tau) \uparrow \infty$ then $\lim_{\tau\downarrow 0} \Lambda_\tau^{(t)}(u, h(\tau)) = 0$, for $u = 0$ and ∞ otherwise;

(iii) if $\sqrt{\tau}/h(\tau) \downarrow 0$ then $\lim_{\tau\downarrow 0} \Lambda_\tau^{(t)}(u, h(\tau)) = 0$, for all $u \in \mathbb{R}$.

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The only non-trivial case is (i) and this is the correct "forward time-scale" to use in the analysis.
Even though the limit is zero (clearly not essentially smooth!) a large deviations principle still holds (with a speed $\sqrt{\tau}$).

Small-maturity forward smile: out-the-money options

Proposition (Jacquier-R, 2013), $t > 0$

The following expansion holds for the forward smile for all $k \in \mathbb{R}^*$ as $\tau \downarrow 0$:

$$\sigma^2_{t,\tau}(k) = \begin{cases} \dfrac{v_0(k,t)}{\tau^{1/2}} + \dfrac{v_1(k,t)}{\tau^{1/4}} + o\left(\dfrac{1}{\tau^{1/4}}\right), & \text{if } 4\kappa\theta \neq \xi^2, \\ \dfrac{v_0(k,t)}{\tau^{1/2}} + \dfrac{v_1(k,t)}{\tau^{1/4}} + v_2(k,t) + v_3(k,t)\tau^{1/4} + o\left(\tau^{1/4}\right), & \text{if } 4\kappa\theta = \xi^2. \end{cases}$$

- Compare with the $t = 0$ case: $\sigma^2_{0,\tau}(k) = \sigma^2_0(k) + a(k)\tau + o(\tau)$, when $k \neq 0$.
- Both $v_0(\cdot, t)$ and $v_1(\cdot, t)$ are positive, even functions and correlation-independent quantities so that for small maturities the Heston forward smile becomes symmetric (in log-strikes) around the ATM point.
- Consequently, if one believes that the small-maturity forward smile should be downward sloping (similar to the spot smile) then the Heston model should not be chosen.
- Bühler (2002): *'Heston implied forward volatility: short skew becomes U-shaped, which is inconsistent with observations.'*

Small-maturity forward smile: at-the-money options

In the at-the-money case the small-maturity limit is well-defined; in fact the zeroth order term below holds for any stochastic volatility model (S, V).

At-the-money case $k = 0$, $t > 0$.

As $\tau \downarrow 0$,

$$\sigma_{t,\tau}(0) = \begin{cases} \mathbb{E}(\sqrt{V_t}) + o\left(1\right), & \text{if } 4\kappa\theta \leq \xi^2, \\ \mathbb{E}(\sqrt{V_t}) + \Delta_0(t)\tau + o(\tau), & \text{if } 4\kappa\theta > \xi^2. \end{cases}$$

Why is there a difference with out-of-the-money options? : At-the-money call options decay algebraically to zero whereas out-of-the-money options decay exponentially to zero as $\tau \downarrow 0$.

The good correlation regime: large-maturity Heston

Set $V(u) \equiv \lim_{\tau\uparrow\infty} \tau^{-1} \log \mathbb{E}\left(\mathrm{e}^{uX_\tau^{(t)}}\right)$. Under what conditions is V essentially smooth on its effective domain? This is most easily stated as a condition on the Heston correlation:

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Lemma: If $\rho_-(t) \leq \rho \leq \min(\rho_+(t), \kappa/\xi)$ then V is essentially smooth on its effective domain.

Here $\rho_\pm(t)$ are functions of the Heston model parameters. Also $-1 \leq \rho_-(t) < 0$ and $0 < \min\left(\rho_+(t), \kappa/\xi\right) \leq 1$ i.e. this defines an interval around zero which we call the **good correlation regime**.

The good correlation regime: large-maturity Heston

The following result is a special case of a much more general result in our paper:

Proposition (Jacquier-R, 2013)

If $\rho_-(t) \leq \rho \leq \min(\rho_+(t), \kappa/\xi)$ then for all $k \in \mathbb{R}$ as $\tau \uparrow \infty$:

$$\sigma^2_{t,\tau}(k\tau) = v_0^\infty(k) + v_1^\infty(k,t)\tau^{-1} + v_2^\infty(k,t)\tau^{-2} + \mathcal{O}(\tau^{-3}),$$

where $v_0^\infty(\cdot)$, $v_1^\infty(\cdot, t)$ and $v_2^\infty(\cdot, t)$ are continuous functions on $\mathbb{R}$.

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Natural Conjecture: (eg. Balland(2006)) the limiting large-maturity forward smile is the same as the limiting large-maturity spot smile (e.g. no dependence on the forward-start date t).

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Corollary: If $\rho_-(t) \leq \rho \leq \min(\rho_+(t), \kappa/\xi)$ then the conjecture holds.

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Corollary: If $\rho_-(t) \leq \rho \leq \min(\rho_+(t), \kappa/\xi)$ then the conjecture holds.

What happens outside these bounds? Let us consider the practically relevant case $\rho < \rho_-(t)$...

The asymmetric correlation regime: $\rho < \rho_-(t)$

Proposition (Jacquier-R, 2014), $t > 0$, $\rho < \rho_-(t)$, $\tau \uparrow \infty$

(i) For all $k < k^*(t)$: $\sigma^2_{t,\tau}(k\tau) = v_0^\infty(k,t) + v_1^\infty(k,t)/\tau + v_2^\infty(k,t)/\tau^2 + \mathcal{O}\left(1/\tau^3\right)$,

(ii) For $k = k^*(t)$: $\sigma^2_{t,\tau}(k\tau) = \widetilde{v}^\infty_{0,+}(k,t) + \widetilde{v}^\infty_{1,+}(t)/\tau^{2/3} + o\left(1/\tau^{2/3}\right)$,

(iii) For all $k > k^*(t)$:

$$\sigma^2_{t,\tau}(k\tau) = \begin{cases} v^\infty_{0,+}(k,t) + \dfrac{v^\infty_{1,+}(k,t)}{\tau^{1/2}} + o\left(\dfrac{1}{\tau^{1/2}}\right), & \text{if } 4\kappa\theta \neq \xi^2, \\ v^\infty_{0,+}(k,t) + \dfrac{v^\infty_{1,+}(k,t)}{\tau^{1/2}} + \dfrac{v^\infty_{2,+}(k,t)}{\tau} + \mathcal{O}\left(\dfrac{1}{\tau^{3/2}}\right), & \text{if } 4\kappa\theta = \xi^2. \end{cases}$$

- The critical strike $k^*(t)$ is a function of the Heston model parameters and is always greater than the at-the-money.
- This new regime adds convexity to the right-wing of the forward smile which is due to extreme positive moment explosions of the forward price process.

The asymmetric correlation regime

Here we compare the large-maturity spot smile and large-maturity forward smile in the asymmetric regime using our zeroth order asymptotics. At the critical strike we see the divergence between the forward smile and spot smile as we enter the new regime.

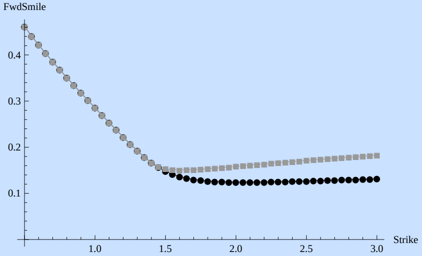

Figure: Parameters are $v = \theta = 0.1, \kappa = 2, \xi = 1, \rho = -0.9$. Circles plot spot smile ($t = 0, \tau = 2$) and squares plots forward smile ($t = 0.5, \tau = 2$). $\rho_- \approx -0.63$ and the critical strike is $\mathrm{e}^{2k^*(0.5)} \approx 1.41$.

The asymmetric correlation regime

- This asymmetric feature of the Heston forward smile is a fundamental property of the model — not only for large-maturities.
- Bergomi (2004): "...the increased convexity (of the forward smile) with respect to today's smile is larger for $k > 0(ATM)$ then for $k < 0$...this is specific to the Heston model."

Conclusions

We studied small and large-maturity forward smile asymptotics in the Heston model. We then used these results to gain insight into some dynamical properties of the model.

- In the analysis we identified a number of cases of degenerate large deviations behaviour.
- These cases uncovered fundamental dynamical properties of the model: It is in fact the analysis of these pathological cases that we learnt the most about the model!
- Finally, these degenerate cases were related back to important empirical observations made by practitioners.

Getting some intuition: what is the cause?

Recall spot Heston:

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}V_t\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t, &\quad X_0 = 0, \\ \mathrm{d}V_t &= \kappa(\theta - V_t)\mathrm{d}t + \xi\sqrt{V_t}\mathrm{d}Z_t, &\quad V_0 = v > 0, \end{aligned}$$

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Define $X_\tau^{(t)} := X_{t+\tau} - X_t$, then

$$\begin{aligned} \mathrm{d}X_\tau^{(t)} &= -\frac{1}{2}V_\tau^{(t)}\mathrm{d}\tau + \sqrt{V_\tau^{(t)}}\mathrm{d}W_\tau, \qquad && X_0^{(t)} = 0, \\ \mathrm{d}V_\tau^{(t)} &= \kappa(\theta - V_\tau^{(t)})\mathrm{d}\tau + \xi\sqrt{V_\tau^{(t)}}\mathrm{d}Z_\tau, && V_0^{(t)} \sim V_t. \end{aligned}$$

Getting some intuition: what is the cause?

Recall spot Heston:

$$\begin{aligned} \mathrm{d}X_t &= -\frac{1}{2}V_t\mathrm{d}t + \sqrt{V_t}\mathrm{d}W_t, \qquad X_0 = 0, \\ \mathrm{d}V_t &= \kappa(\theta - V_t)\mathrm{d}t + \xi\sqrt{V_t}\mathrm{d}Z_t, \quad V_0 = v > 0, \end{aligned}$$

Define $X_\tau^{(t)} := X_{t+\tau} - X_t$, then

$$\begin{aligned} \mathrm{d}X_\tau^{(t)} &= -\frac{1}{2}V_\tau^{(t)}\mathrm{d}\tau + \sqrt{V_\tau^{(t)}}\mathrm{d}W_\tau, \qquad X_0^{(t)} = 0, \\ \mathrm{d}V_\tau^{(t)} &= \kappa(\theta - V_\tau^{(t)})\mathrm{d}\tau + \xi\sqrt{V_\tau^{(t)}}\mathrm{d}Z_\tau, \quad V_0^{(t)} \sim V_t. \end{aligned}$$

Pricing Fwd-start options:

$$\mathbb{E}_0(\mathrm{e}^{X_\tau^{(t)}} - \mathrm{e}^k)^+ = \mathbb{E}_0\left\{\mathbb{E}\left[(\mathrm{e}^{X_\tau^{(t)}} - \mathrm{e}^k)^+ | V_t\right]\right\}$$

Intuition for explosion: Initial variance is a random variable. Out-the-money options are convex in volatility and we are squeezing a variance distribution into a single number as $\tau \downarrow 0$.